EFFECT OF FREE CONVECTION ON THE INSTABILITY OF A PLANE CRYSTALLIZATION FRONT

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The article considers the problem of free convection under conditions of phase transition. The method of small perturbations is used to study the stability of the crystallization front. Critical values of the parameters and calculations of the region of instability are given.

1. Statement of Problem. Let us consider a system consisting of a liquid and a solid phase. In an unperturbed state, the liquid phase occupies the region $0<z<l$, and the solid phase the region $l<z<H$. The plane $z=l$ is the phase transition interface and has a constant temperature, equal to the melting point T_*. In the plane $z=0$, the temperature T_0 is given.

It follows from the thermal conductivity equation that in an equilibrium state for the liquid phase

$$dT / dz = (T_* - T_0) / l = \beta = \text{const} \tag{1.1}$$

For the solid phase, taking account of (1.1) and of the condition of the equality of the heat fluxes at the phase interface, we obtain the result that in the equilibrium state

$$dT_1 / dz = \sigma\beta, \quad \sigma = \varkappa / \varkappa_1 \tag{1.2}$$

where T and T_1 are the temperatures of the liquid and solid phases; $\varkappa$ and $\varkappa_1$ are the thermal conductivity coefficients.

Let the quantities characterizing the state under consideration, including the crystallization front, undergo small perturbations. The equations of the perturbed state for the solid and liquid phases can be linearized and written in dimensionless form. If, as characteristic values of the velocity, the time, the length, and the temperature, we take, respectively, χ_0/l, l^2/ν, l, T_0-T_*, then the Prandtl number Pr and Rayleigh number Ra are

$$\text{Pr} = \nu / \chi_0, \quad \text{Ra} = g\alpha_0 (T_0 - T_*) l^3 / \chi_0 \nu$$

Here ν is the kinematic viscosity; χ_0 is the coefficient of thermal diffusivity in the liquid phase; g is the acceleration due to gravity; and α_0 is the coefficient of volumetric expansion of the liquid phase.

In view of the homogeneity of the problem with respect to the horizontal coordinates x and y and the time t, the solution of the system of equations of the perturbed state may be sought in the wave form

$$\varphi = \Phi(\zeta) \exp i (m\xi + n\eta - \omega\tau)$$

Here φ is any one of the dimensionless characteristics of the perturbed flow; $\xi=x/l$, $\eta=y/l$, $\zeta=z/l$, $\tau=t\nu/l^2$.

After transformations, the system of equations describing the perturbed state in the liquid phase can be reduced [1] to a single equation with respect to the amplitude of the temperature perturbation Θ

$$\left(\frac{d^2}{d\zeta^2} - k^2\right)\left(\frac{d^2}{d\zeta^2} - k^2 + i\omega\right)\left(\frac{d^2}{d\zeta^2} - k^2 + \text{Pr}\, i\omega\right)\Theta + k^2 \text{Ra}\,\Theta = 0 \tag{1.3}$$
$$(k^2 = m^2 + n^2)$$

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In this case, the amplitudes of the velocity perturbations U, V, W are expressed in terms of the perturbation of the temperature in the following manner:

$$W(\zeta) = -\left(\frac{d^2}{d\zeta^2} - k^2 + \Pr i\omega\right)\Theta, \qquad mU + nV = i\frac{dW}{d\zeta} \tag{1.4}$$

The equation for the amplitude of the temperature perturbation in the solid phase, Θ_1, has the form

$$\left(\frac{d^2}{d\zeta^2} - k^2 + \Pr q^{-1} i\omega\right)\Theta_1 = 0, \qquad q = \chi_1 / \chi_0 \tag{1.5}$$

where χ_1 is the coefficient of thermal diffusivity in the solid phase.

We write the boundary conditions which must be satisfied by the amplitude of the perturbations. The surface $z = 0$ is assumed to be rigid with a given temperature T_0; therefore, the perturbations of the temperature and the velocity revert to zero at this surface. Taking account of (1.4) we will have

$$W = \frac{dW}{d\zeta} = \Theta = \frac{d^2\Theta}{d\zeta^2} = \frac{d}{d\zeta} = \left(\frac{d}{d\zeta^2} - k^2 + \Pr i\omega\right)\Theta = 0 \quad \text{at} \quad \zeta = 0 \tag{1.6}$$

At the upper boundary of the solid phase we shall, in what follows, consider two forms of boundary conditions:

a) the condition of constancy of the temperature

$$\Theta_1 = 0 \quad \text{at} \quad \zeta = a = H / l \tag{1.7}$$

b) the condition of constancy of the heat flux

$$d\Theta_1 / d\zeta = 0 \quad \text{at} \quad \zeta = a \tag{1.8}$$

Let us consider the conditions at the phase transition interface. Let the equation of the perturbed phase-transition surface have the form

$$\zeta = 1 + Z\exp i\,(m\xi + n\eta - \omega\tau) \tag{1.9}$$

Since the temperature at the phase interface remains constant, equal to the melting temperature, the perturbations of the temperature in the liquid and solid phases must revert to zero at the surface (1.9). Decomposing the temperature of the perturbed state in the neighborhood $\zeta = 1$ into a Taylor series, we find

$$\Theta = Z, \ \Theta_1 = \sigma Z \quad \text{at} \quad \zeta = 1 \tag{1.10}$$

The condition for adhesion requires the reversion to zero of the tangential component of the velocity at the surface (1.9). Therefore

$$\frac{dW}{d\zeta} = \frac{d}{d\zeta}\left(\frac{d^2}{d\zeta^2} k^2 + \Pr i\omega\right)\Theta = 0 \quad \text{at} \quad \zeta = 1 \tag{1.11}$$

At the perturbed-phase transition surface, the laws of conservation of the flows of mass and energy must also be satisfied [2]:

$$\rho_0 (D_n - u_n) = \rho_1 D_n, \quad \varkappa_1 \frac{\partial T_1'}{\partial z} - \varkappa \frac{\partial T'}{\partial z} = \rho_0 \lambda D_n$$

where D_n is the normal velocity of the phase-transition surface; λ is the specific heat of fusion; ρ_0 and ρ_1 are the densities of the liquid and solid phases at the melting temperature; and T_1' and T' are perturbations of the temperature.

In dimensionless form, for the amplitude of the characteristic curve of the perturbed state, taking account of (1.9), these conditions are written in the following manner:

$$W = -\left(\frac{d^2}{d\zeta^2} - k^2 + Pr\ i\omega\right)\Theta = -i\omega r Z \qquad \left(r = \frac{\rho_0 - \rho_1}{\rho_0}\right) \quad \text{at} \quad \zeta = 1 \tag{1.12}$$

$$\frac{1}{\sigma}\frac{d\Theta_1}{d\zeta} - \frac{d\Theta}{d\zeta} = -i\omega R Z \qquad \left(R = \frac{\rho\lambda\nu}{\varkappa(T_0 - T_*)}\right. \quad \text{at} \quad \zeta = 1 \tag{1.13}$$

Thus, the investigation of the stability of the plane phase transition surface is brought down to the problem of the eigenvalues for Eqs. (1.3) and (1.5), with the boundary conditions (1.6)-(1.8) and (1.10)-(1.13).

2. Proof of the Real Nature of the Eigenvalues. Equation (1.3) can be written in the form of a system of two equations

$$k^2 \operatorname{Ra} \Theta = \left(\frac{d^2}{d\zeta^2} - k^2 + i\omega\right)\left(\frac{d^2}{d\zeta^2} - k^2\right) W \tag{2.1}$$
$$W = -\left(\frac{d^2}{d\zeta^2} - k^2 + i\omega \operatorname{Pr}\right)\Theta$$

To prove the real nature of the eigenvalues, we use the method of Pellow and Southwell [1]. We multiply the first equation of (2.1) by $\bar{W}$, and the second by $\bar{\Theta}$, and then integrate between the limits from 0 to 1. The bar above the symbol denotes complex-conjugate quantities. Using boundary conditions (1.6) and (1.11), we obtain

$$\int_0^1 W\bar{\Theta}d\zeta = -\Theta'(1)\bar{\Theta}(1) + (J_1^2 + k^2J_0^2) - i\omega \operatorname{Pr} J_0^2 \tag{2.2}$$
$$k^2 \operatorname{Ra} \int_0^1 \Theta\bar{W}d\zeta = G'(1)\bar{W}(1) + I_2^2 + 2k^2I_1^2 + k^4I_0^2 - i\omega(I_1^2 + k^2I_0^2)$$

Here

$$G = \frac{d^2W}{d\zeta^2} - k^2W, \quad I_2^2 = \int_0^1 |W''|^2 d\zeta, \quad I_1^2 = \int_0^1 |W'|^2 d\zeta$$
$$I_0^2 = \int_0^1 |W|^2 d\zeta, \quad J_1^2 = \int_0^1 |\Theta'|^2 d\zeta, \quad J_0^2 = \int_0^1 |\Theta|^2 d\zeta$$

Analogously, multiplying Eq. (1.5) by $\bar{\Theta}_1$ and integrating between the limits from 1 to a, using (1.7), (1.8), and (1.10), we obtain

$$\Theta_1'(1)\bar{Z} = \frac{1}{\sigma}(-J_{11}^2 - k^2J_{10}^2 + i\omega q^{-1} \operatorname{Pr} J_{10}^2) \tag{2.3}$$

where J_{10}^2, J_{11}^2 are integrals, analogous to J_0^2 and J_1^2. Substituting (2.3) into the heat-balance equation, we can determine the value of Θ_1' at $\zeta=1$, which is then substituted into the first of the relationships (2.2). The value of $\bar{\Theta}$ at $\zeta=1$ can be found using (1.10).

Using the dimensionless equations of the perturbed state and the second of the equalities (1.4), for the amplitude of the perturbation of the pressure P we obtain

$$\mathrm{P}(1) = i\omega k^{-2}W'(1) + \operatorname{Pr}k^{-2}G'(1) \tag{2.4}$$

From boundary condition (1.11) it follows that

$$W'(1) = 0$$

The amplitude of the perturbation of the pressure at the surface may be evaluated as the perturbation of the hydrostatic pressure, due to the replacement of a liquid element by a solid element

$$\mathrm{P}(1) \approx -r\operatorname{Fr}^{-2}Z, \quad \operatorname{Fr}^2 = \chi_0^2/gl^3$$

whence

$$G'(1) = -\frac{rk^2}{\operatorname{Pr}\operatorname{Fr}^2}Z \tag{2.5}$$

From condition (1.12)

$$\bar{W}(1) = i\bar{\omega}r\bar{Z} \tag{2.6}$$

Substituting (2.5) and (2.6) into the second equation of (2.2), and noting that the left- and right-hand parts of the first and second equations of (2.2), with an accuracy up to the factor k^2Ra, are complex-conjugate, we obtain

$$-\frac{k^2}{\operatorname{Pr}\operatorname{Fr}^2}i\bar{\omega}|Z|^2r^2 + M^2 - i\omega N^2 = -k^2 \operatorname{Ra}\Big[\frac{1}{\sigma^2}\Big(-K^2 - i\bar{\omega}q^{-1}\operatorname{Pr}J_{10}^2\Big) - \tag{2.7}$$
$$- i\bar{\omega}R|Z|^2 - L^2 - i\bar{\omega}\operatorname{Pr}J_0^2\Big]$$

Here

$$M^2 = I_2^2 + 2k^2 I_1^2 + k^4 I_0^2, N^2 = I_1^2 + k^2 I_0^2$$
$$K^2 = I_{11}^2 + k^2 J_{10}^2, \quad L^2 = J_1^2 + k^2 J_0^2$$

We set $s = s_r + is_i = -i\omega$ and equate the real and imaginary parts in (2.7)

$$[N^2 + k^2 \mathrm{Pr}^{-1} \mathrm{Fr}^{-2} r^2 \mid Z \mid^2 + k^2 \mathrm{Ra}\,(q^{-1}\sigma^{-2} J_{10}^2 + R \mid Z \mid^2 + J_0^2)] s_i = 0$$
$$[N^2 - k^2 \mathrm{Pr}^{-1} \mathrm{Fr}^{-2} r^2 \mid Z \mid^2 - k^2 \mathrm{Ra}\,(q^{-1}\sigma^{-2} J_{10}^2 + R \mid Z \mid^2 + J_0^2)] s_r = k^2 \mathrm{Ra}\,(L^2 + \sigma^{-2} K^2) - M^2 \tag{2.8}$$

An unstable state corresponds to $s_r > 0$, and a stable state to $s_r < 0$.

The expression in square brackets in the first equation of (2.8) cannot revert to zero in the case when $\mathrm{Ra} > 0$.

Therefore, $s_i = 0$ ($\omega_r = 0$); s is a real value, and the transition from a stable state to an unstable state takes place with $s = 0$ or $\omega = 0$.

3. Determination of the Critical Rayleigh Numbers and of the Eigenvalues. The value of the parameter R for crystalline solids is usually very large. It follows from the boundary condition (1.13) that the eigenvalue ω is small; in the contrary case there would be very large temperature gradients. In view of this, we shall seek the solution of Eqs. (1.3), (1.5) with the boundary conditions (1.6)-(1.13), using series in terms of a small parameter:

$$\Theta = \Theta_0 + \omega\Theta_1 + \ldots, \quad \Theta_1 = \Theta_{10} + \omega\Theta_{11} + \ldots \tag{3.1}$$

In this case, in the boundary condition (1.13) we retain the term with the product $R\omega$, in view of the large value of the parameter R. This permits determining the eigenvalue ω even in the zero approximation. The critical value of the Rayleigh number corresponds to $\omega = 0$; therefore, the selected method of approximate analysis also permits determining the critical Rayleigh number.

For the zero approximation, we have the following system of equations:

$$(d^2 / d\zeta^2 - k^2)^3 \Theta_0 + k^2 \mathrm{Ra}\Theta_0 = 0, \quad (d^2 / d\zeta^2 - k^2)\Theta_{10} = 0 \tag{3.2}$$

The values of Θ_0 and Θ_{10} must satisfy the boundary conditions (in what follows, for simplicity we omit the subscript 0)

$$\Theta = \frac{d^2\Theta}{d\zeta^2} = \frac{d}{d\zeta}\left(\frac{d^2}{d\zeta^2} - k^2\right)\Theta = 0 \quad \text{at} \quad \zeta = 0 \tag{3.3}$$

$$\Theta = Z, \quad \Theta_1 = \sigma^{-1} Z, \quad d^2\Theta / d\zeta^2 = k^2 Z \quad \text{at} \quad \zeta = 1 \tag{3.4}$$
$$\frac{d}{d\zeta}\left(\frac{d^2}{d\zeta^2} - k^2\right)\Theta = 0, \quad \sigma\frac{d\Theta_1}{d\zeta} - \frac{d\Theta}{d\zeta} = -i\omega RZ$$

$$\Theta_1 = 0 \quad \text{at} \quad \zeta = a \tag{3.5}$$

or

$$d\Theta_1 / d\zeta = 0 \tag{3.6}$$

From the second equation of (3.2) and the boundary conditions for Θ, with $\zeta = a$, for cases a) and b), respectively, we obtain

$$\Theta_1 = \frac{Z}{\sigma}\frac{\operatorname{sh} k(a - \zeta)}{\operatorname{sh} k(a - 1)} \tag{3.7}$$

or

$$\Theta_1'(1) = -kZ\sigma^{-1}\operatorname{cth} k(a - 1) \tag{3.8}$$

and

$$\Theta_1 = \frac{Z}{\sigma}\frac{\operatorname{ch} k(a - \zeta)}{\operatorname{ch} k(a - 1)} \tag{3.9}$$

in which case

$$\Theta_1'(1) = -kZ\sigma^{-1}\operatorname{th} k(a - 1) \tag{3.10}$$

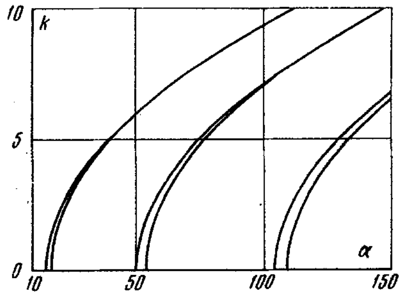

Fig. 1

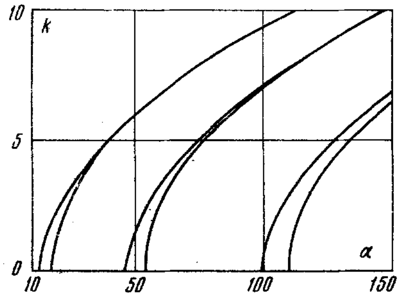

Fig. 2

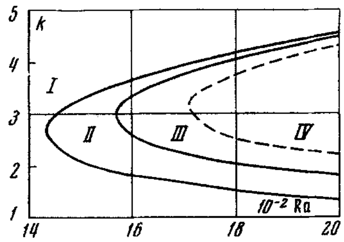

Fig. 3

Substituting (3.8) and (3.10) into the last boundary condition of (3.4), we obtain

$$\Theta'(1) = i\omega ZR - kZ\operatorname{cth} k(a-1) \tag{3.11}$$

$$\Theta'(1) = i\omega ZR - kZ\operatorname{th} k(a-1) \tag{3.12}$$

The general solution of the first equation of (3.2) has the form

$$\Theta = a_1 e^{\lambda_1\xi} + a_2 e^{-\lambda_1\xi} + a_3 e^{\lambda_2\xi} + a_4 e^{-\lambda_2\xi} + a_5 e^{\lambda_3\xi} + a_6 e^{-\lambda_3\xi} \tag{3.13}$$

Here λ_i (i = 1, 2, 3) are the roots of the characteristic equation

$$(\lambda^2 - k^2)^3 + k^2\mathrm{Ra} = 0 \tag{3.14}$$

$$\lambda_1 = \sqrt{k^2 - \alpha}, \quad \lambda_{2,3} = \sqrt{k^2 + {}^1/_2\alpha(1 \pm i\sqrt{3})}, \quad \alpha = \sqrt[3]{k^2\mathrm{Ra}}$$

Substituting (3.13), (3.14) into boundary conditions (3.3), (3.4), and (3.11) or (3.12), we obtain a system of equations for determining the six arbitrary coefficients and the amplitude of the shift in the crystallization front Z. Since the system of equations obtained is found to be homogeneous, its determinant should revert to zero. We obtain linear equations for determining the frequency ω in cases a) and b), respectively:

$$i\omega = \frac{1}{R}\left[k\operatorname{cth} k(a-1) + \frac{B}{A}\right] \tag{3.15}$$

$$i\omega = \frac{1}{R}\left[k\operatorname{th} k(a-1) + \frac{B}{A}\right] \tag{3.16}$$

A and B are the minors of the determinant; $A = \varphi(1)$, $B = \varphi'(1)$, where

$$\varphi(\xi) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1^2 & x_1^2 & x_2^2 & x_2^2 & x_3^2 & x_3^2 \\ \lambda_1 x_1^2 & -\lambda_1 x_1^2 & \lambda_2 x_2^2 & -\lambda_2 x_2^2 & \lambda_3 x_3^2 & -\lambda_3 x_3^2 \\ e^{\lambda_1\xi} & e^{-\lambda_1\xi} & e^{\lambda_2\xi} & e^{-\lambda_2\xi} & e^{\lambda_3\xi} & e^{-\lambda_3\xi} \\ x_1^2 e^{\lambda_1} & x_1^2 e^{-\lambda_1} & x_2^2 e^{\lambda_2} & x_2^2 e^{-\lambda_2} & x_3^2 e^{\lambda_3} & x_3^2 e^{-\lambda_3} \\ \lambda_1 x_1^2 e^{\lambda_1} & -\lambda_1 x_1^2 e^{-\lambda_1} & \lambda_2 x_2^2 e^{\lambda_2} & -\lambda_2 x_2^2 e^{-\lambda_2} & \lambda_3 x_3^2 e^{\lambda_3} & -\lambda_3 x_3^2 e^{-\lambda_3} \end{vmatrix} \tag{3.17}$$

$$x_i^2 = \lambda_i^2 - k^2$$

The determinant (3.17) can be put in the following form:

$$\varphi(\xi) = \frac{8}{\lambda_1 x_1^2}\begin{vmatrix} \lambda_1 x_1^2\Phi_2 - \lambda_2 x_2^2\Phi_1 & \Lambda_{12} & \lambda_1 x_1^2\Phi_3 - \lambda_3 x_3^2\Phi_1 \\ x_1^2 x_2^2(\lambda_1 S_2 - \lambda_2 S_1) & \Lambda_{22} & x_1^2 x_3^2(\lambda_1 S_3 - \lambda_3 S_1) \\ \lambda_1\lambda_2 x_1^2 x_2^2 C_{21} & \Lambda_{32} & \lambda_1\lambda_3 x_1^2 x_3^2 C_{31} \end{vmatrix} \tag{3.18}$$

where

$$\Lambda_{12} = x_1^2\Psi_{23} + x_2^2\Psi_{31} + x_3^2\Psi_{12}, \quad \Lambda_{22} = x_2^2 x_3^2 C_{32} - x_1^2 x_3^2 C_{13}$$

$$\Lambda_{32} = x_2^2 x_3^2(\lambda_3 S_3 - \lambda_2 S_2) + x_1^2 x_2^2(\lambda_2 S_2 - \lambda_1 S_1) + x_3^2 x_1^2(\lambda_1 S_1 - \lambda_3 S_3)$$

$$S_i = \operatorname{sh}\lambda_i, \quad C_i = \operatorname{ch}\lambda_i, \quad C_{ij} = C_i - C_j, \quad S_{ij} = S_i - S_j$$

$$\Phi_{ij} = \Phi_i - \Phi_j, \quad \Psi_{ij} = \Psi_i - \Psi_j$$

$$\Phi_i = \begin{cases} \operatorname{sh}\lambda_i & \text{for the minor } A, \\ \lambda_1\operatorname{ch}\lambda_i & \text{for the minor } B, \end{cases} \quad \Psi_i = \begin{cases} \operatorname{ch}\lambda_i & \text{for the minor } A \\ \lambda_1\operatorname{sh}\lambda_1 & \text{for the minor } B \end{cases}$$

Substituting the functions Φ_i and Ψ_i and expanding the determinant (3.18), we find

$$A = 48\alpha^3 [\lambda_1\lambda_2 S_3 x_3^2 (1 - C_1C_2) + \lambda_1\lambda_3 S_2 x_2^2 (1 - C_1C_3) + \lambda_2\lambda_3 S_1 x_1^2 (1 - C_2C_3)] \tag{3.19}$$

$$B = 24\alpha^3 [2 (C_1x_1^2 + C_2x_2^2 + C_3x_3^2) \lambda_1\lambda_2\lambda_3 + S_1S_2 (\lambda_1^2 + \lambda_2^2) C_3\lambda_3x_3^2 + S_1S_3 (\lambda_1^2 + \lambda_3^2) C_2\lambda_2x_2^2 + S_2S_3 (\lambda_2^2 + \lambda_3^2) C_1\lambda_1x_1^2]$$

In the transformation of (3.18) into (3.19), the properties of the roots of Eq. (3.14) were used.

The quantities x_2^2 and x_3^2, λ_2 and λ_3, C_2 and C_3, S_2 and S_3 are complex-conjugate. The root λ_1 is real at $k^2-\alpha>0$ and purely imaginary at $k^2-\alpha<0$. Correspondingly to this, $\mathrm{sh}\lambda_1=S_1$ can be real or purely imaginary. It is easily shown that the minors A and B, in accordance with the value of λ_1, are simultaneously real or purely imaginary. Actually, the expressions for A and B can be transformed in the following manner:

$$A = 48\alpha^4 [\lambda_1 \{[(\gamma + \sqrt{3}\delta)\,\mathrm{sh}\,\gamma \cos\delta + (\delta - \sqrt{3}\gamma)\sin\delta\,\mathrm{ch}\,\gamma] - \frac{1}{2}\mathrm{ch}\,\lambda_1 [(\gamma + \sqrt{3}\delta)\,\mathrm{sh}\,2\gamma + (\delta - \sqrt{3}\gamma)\sin 2\delta]\} + \mathrm{sh}\,\lambda_1 (\gamma^2 + \delta^2)(\mathrm{sh}^2\gamma - \sin^2\delta)] \tag{3.20}$$

$$B = 24\alpha^4 [2\lambda_1 (\gamma^2 + \delta^2)(\mathrm{ch}\,\gamma\cos\delta - \sqrt{3}\,\mathrm{sh}\,\gamma\sin\delta - \mathrm{ch}\,\lambda_1) + {}^1/_2\,\mathrm{sh}\,\lambda_1 \{[(2k^2 + \alpha)\gamma + \sqrt{3}\,\delta(\alpha - 2k^2)]\,\mathrm{sh}\,2\gamma - [\sqrt{3}\gamma(\alpha - 2k^2) - \delta(\alpha + 2k^2)]\sin 2\delta\} - \lambda_1\,\mathrm{ch}\,\lambda_1 (2k^2 + \alpha)(\mathrm{ch}^2\gamma - \cos^2\delta)] \tag{3.21}$$

where

$$\gamma = \mathrm{Ra}\lambda_2,\ \delta = \mathrm{Im}\lambda_2$$

The factors with the expressions for λ_1 and $\mathrm{sh}\lambda_1$ are real; therefore, when λ_1 is real, the expressions for A and B are real, while when λ_1 is purely imaginary, they are purely imaginary, and their ratio is always real. Thus, the results obtained in Sec. 2 with respect to the existence of the eigenvalue $s=-i\omega$ is confirmed.

With large values of the parameter α in comparison with k^2,

$$\alpha = \sqrt[3]{k^2\,\mathrm{Ra}} \gg k^2, \qquad \mathrm{Ra} \gg k^4$$

the minors A and B have the following asymptotic expressions:

$$A = 48\alpha^5 i\Big(-\sin\sqrt{\alpha} + \frac{1}{2}\sin 2\sqrt{\alpha} - \sin\frac{\sqrt{\alpha}}{2}\mathrm{ch}\frac{\sqrt{3\alpha}}{2} + \sqrt{3}\,\mathrm{ch}\frac{\sqrt{3\alpha}}{2}\cos\frac{\sqrt{\alpha}}{2} + \frac{1}{2}\mathrm{ch}\sqrt{3\alpha}\sin\sqrt{\alpha} - \frac{\sqrt{3}}{2}\mathrm{sh}\sqrt{3\alpha}\cos\sqrt{\alpha}\Big) \tag{3.22}$$

$$B = 24\alpha^5\sqrt{\alpha}\, i\,[-2\cos\sqrt{\alpha} + 0.5\cos 2\sqrt{\alpha} + 0.5\,(\sqrt{3}\sin\sqrt{\alpha}\,\mathrm{sh}\sqrt{3\alpha} - \mathrm{ch}\sqrt{3\alpha}\cos\sqrt{\alpha}) + 2\,(\mathrm{ch}\,0.5\sqrt{3\alpha}\cos 0.5\sqrt{\alpha} - \sqrt{3}\,\mathrm{sh}\,0.5\sqrt{3\alpha}\sin 0.5\sqrt{\alpha})]$$

For $\alpha\gg1$, the ratio B/A has the form

$$\frac{B}{A} = -\frac{\sqrt{\alpha}}{2}\frac{\cos(\sqrt{\alpha}+\varepsilon)}{\sin(\sqrt{\alpha}-\varepsilon)} \approx -\frac{\sqrt{\alpha}}{2}\frac{\cos(\sqrt{\alpha}+\pi/3)}{\sin(\sqrt{\alpha}-\pi/3)} \tag{3.23}$$

$$\mathrm{tg}\,\varepsilon = \sqrt{3}\,\mathrm{th}\sqrt{3\alpha} \approx \sqrt{3} = \mathrm{tg}\,\pi/3$$

It follows from expressions (3.15) and (3.16) that instability arises in narrow regions of values of the parameter α, in which $\cos(\sqrt{\alpha}+\pi/3)$ and $\sin(\sqrt{\alpha}-\pi/3)$ have identical signs:

$$\pi n + \pi/6 < \sqrt{\alpha} < \pi n + \pi/3 \tag{3.24}$$

The regions of instability in the plane (α, k) at $a=1.2$, calculated in a digital computer using formulas (3.15), (3.16), (3.20), and (3.21), are given in Figs. 1 and 2. Figure 1 corresponds to the case when the temperature does not vary at the outer surface of the solid phase. Figure 2 corresponds to the case when the heat flux at the outer surface of the solid phase is given. As is evident from the curves, the regions of instability at $k\ll1$ are in good agreement with the asymptotic formula (3.24). With an increase in the value of α, the regions of instability are broadened, while with an increase in the value of k they are narrowed, and are drawn out to a line. In this case, it must be borne in mind that, as follows from the determination of α (3.14), the points $k=0$ and $\alpha\neq0$ on Figs. 1 and 2 correspond to a Rayleigh number $\mathrm{Ra}=\alpha^3/k^2=\infty$.

Figure 3 gives curves of neutral stability, corresponding to $\omega=0$ in the plane (Ra, k). The critical values of the Rayleigh number Ra and k at $a=1.2$ are equal to: for case a) Ra = 1558, k = 2.95; for case b) Ra = 1424, k = 2.65. The regions of instability on Fig. 3 are: for case a) region III; for case b) regions II and III.

With an increase in the relative thickness of the layer of solid phase, the effect of the boundary condition at the external surface is less strongly expressed, and the regions of instability in both cases a) and b) come together. The critical values of the parameters are also found to be close. For example, with $a=2$ and $a=6$, $Ra=1490$, $k=2.8$. The analysis carried out shows that developing convection leads to an earlier appearance of instability in a plane layer bounded by a crystallization surface, in comparison with a layer bounded by solid fixed walls.

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